ECE Ph.D. PRELIMINARY EXAMINATION SOLUTIONS – SPRING 2002

Prob 1

(1)

(A) Let the potential difference between the plates be V_0. The E-fields in the two regions are E_2 & E_1:

$$V_0 = E_1 d_1 + E_2 d_2.$$

At dielectric interface, E is normal,

$$D_1 = D_2 \iff \varepsilon_1 E_1 = \varepsilon_2 E_2.$$

Thus,

$$E_1 = \frac{V_0}{d_1 + d_2 \, {}^{\varepsilon_1}/_{\varepsilon_2}}.$$

At the ~~dielectric interface~~ top plate, there is a surface-charge density

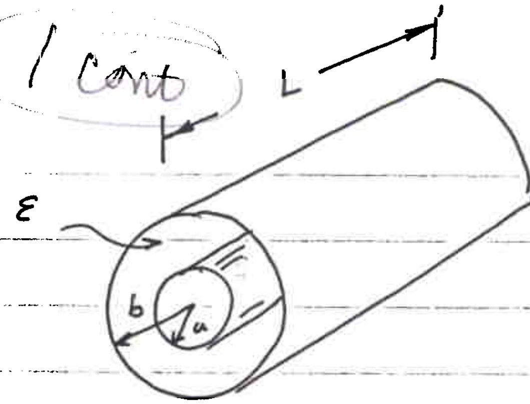
$$\rho_s = D_1 = \varepsilon_1 E_1 = \frac{V_0}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}} = D_2$$

Thus,

$$C = \frac{Q}{V_0} = \frac{\rho_s \overset{\text{area}}{S}}{V_0} = \frac{1}{\frac{d_1}{\varepsilon_1 S} + \frac{d_2}{\varepsilon_2 S}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Prob 1, cont. (1 cont)

(B)



Capacitance

$$C = \frac{2\pi \varepsilon L}{\ln(b/a)}$$

Capacitance / unit length

lim as $a \to \infty$, $b - a = d$ = constant

$$c \sim \lim_{a\to\infty}\left[\frac{1}{2\pi a L} C\right] = \lim_{a\to\infty}\left[\frac{1}{a \ln\left(\frac{a+d}{a}\right)}\right]$$

$$= \lim_{a\to\infty}\left[\frac{1}{a \ln(1+\underbrace{d/a}_{\text{small}})}\right]$$

Note:

$$\ln(1+\Delta) = \underbrace{\ln(1)}_{0} + \left.\frac{d\ln(x)}{dx}\right|_{x=1} \Delta + \cdots$$

$$= \left.\frac{1}{x}\right|_{x=1} \Delta + \cdots = \Delta$$

So

$$c \sim \lim_{a\to\infty}\left[\frac{1}{a\,(d/a)}\right] = 1/d$$

Which is the same as the capacitance / unit area of a parallel plate capacitor with spacing between the plates d.

Answer to Problem 2

2.a) $\lambda_2 = \dfrac{20 \times 10^9}{100 \times 10^6} = 200\ \text{cm} = 2\ \text{m} \Rightarrow \ell_2 = 0.875\lambda$

$\lambda_1 = \dfrac{30 \times 10^9}{100 \times 10^6} = 300\ \text{cm} = 3\ \text{m} \Rightarrow \ell_1 = 1.25\lambda$

$$Z_{in2} = 60\left[\frac{60 - j60 + j60 \tan 1.75\pi}{60 + j(60 - j60)\tan 1.75\pi}\right] = 60\left[\frac{60 - j60 - j60}{60 - j60 - 60}\right]$$

$$\therefore Z_{in2} = 60\left[\frac{60 - j120}{-j60}\right] = 120 + j60\ (\Omega)$$

$$Y_{in2} = \frac{1}{120 + j60} = \frac{120 - j60}{18000} = 6.667 \times 10^{-3} - j3.333 \times 10^{-3}\ (\mho)$$

$$Y_{in\ A-A'} = \frac{1}{150} = 6.667 \times 10^{-3}\ (\mho) = Y_{in2} + Y_{stub}$$ {for max. power transfer at A-A'}

$$\therefore Y_{stub} = +j3.333 \times 10^{-3}\ (\mho)$$

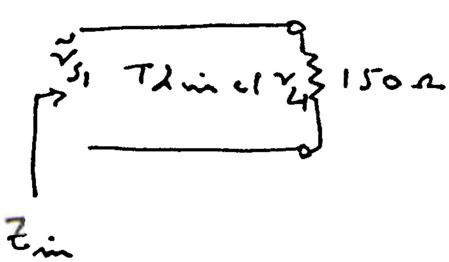
$$\therefore Z_{stub} = -j300\ (\Omega)$$

$$Z_{stub} = 60\left[\frac{0 + j60 \tan(\beta\ell_3)}{60 + j0 \tan \beta\ell_3}\right] = j60 \tan(\beta\ell_3)$$

$$-j300 = j60 \tan(\beta\ell_3) \Rightarrow \tan(\beta\ell_3) = -5$$

$$\therefore \beta\ell_3 = \frac{1.768}{\cancel{1.373}} \Rightarrow \ell_3 = \frac{1.768 \times \lambda_2}{2\pi} = \boxed{0.563\ (\text{m}}$$

b)



$Z_{in} = 150\ \Omega$

$$\tilde{V}_{s_1} = \frac{10 \times 150}{150 + 50} = 7.5\ (\text{V})$$

{voltage at the input of T.line 1

Power delivered to Transmission Line 1 (2 cont)

$$P_{av_1} = \frac{1}{2} Re \left\{ \tilde{V}_{i1} \times \left(\frac{\tilde{V}_{i1}}{Z_{in}} \right)^* \right\}$$

$$\therefore \quad P_{av_1} = \frac{1}{2} Re \left\{ 7.5 \times \frac{7.5}{150} \right\} = 0.1875 \text{ (W)}$$

Since the power absorbed by the short circuit stub equals 0 (W), 0.1875 (W) of power is transferred to Transmission Line 2 since the two transmission lines are matched at A-A'.

$\therefore$ Power delivered to Antenna $\boxed{P_{av} = 0.1875 \text{ (W)}}$ since the transmission lines are lossless.

3)

SOLUTION

a) The air gap power is the power that transfers across the air gap to the rotor. In this case, it is the input power minus the stator copper and core losses.

$$P_{gap} = P_{in} - P_{stator,cu} - P_{core} = 3(480/\sqrt{3})(50)(0.85) -- 1000 - 600 = 33.73kW$$

b) The mechanical power can be written as the air gap power minus the rotor losses.

$$P_{mech} = P_{gap} - P_{rotor,cu} = 33733 - 500 = 33.23kW$$

c) The output power is the mechanical power minus the friction and windage loss.

$$P_{out} = P_{mech} - P_{f\&w} = 33233 - 250 = 33.0kW$$

d) Efficiency is the output power divided by the input power.

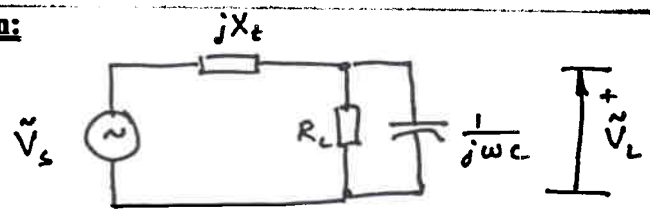
$$\eta = \frac{32984}{3(480/\sqrt{3})(50)(0.85)} = 93.3\%$$

(4)

Problem:

The load connected to the three-phase network consists of the Y-connected impedance load of 20 Ω/phase, and a Δ-connected impedance load of 60 Ω per leg of Δ, connected in parallel. The combined loads are supplied via transmission line of impedance j1 Ohms/phase. Calculate per phase capacitance of the 3-phase capacitor bank that needs to be connected in parallel with the two loads in order to raise the load voltage by 5 percent.

Solution:



$$Z_{1Y} = 20\,\Omega$$

$$Z_{2Y} = \frac{1}{3} Z_{2\Delta} = \frac{1}{3} 60 = 20\,\Omega$$

Problem states that impedance is 60 Ω/phase. Solution is worked with assumption that 60 Ω is per leg of delta.

$$Z_{eq} = R_L = Z_{1Y} \,||\, Z_{2Y} = 10\,\Omega$$

$$Z'_{eq} = R_L \,||\, \frac{1}{j\omega C} = \frac{R_L}{1 + j\omega C R_L}$$

$$\tilde{V}_L(\omega C) = \tilde{V}_s \cdot \frac{Z'_{eq}}{jX_t + Z'_{eq}} = \tilde{V}_s \cdot \frac{\frac{R_L}{1+j\omega C R_L}}{jX_t + \frac{R_L}{1+j\omega C R_L}} = \tilde{V}_s \cdot \frac{R_L}{(R_L - \omega C R_L X_t) + jX_t}$$

$$\left|\frac{\tilde{V}_L(\omega C)}{\tilde{V}_L(0)}\right| = 1.05 = \left|\frac{\tilde{V}_s \cdot \frac{R_L}{(R_L - \omega C R_L X_t) + jX_t}}{\tilde{V}_s \cdot \frac{R_L}{R_L + jX_t}}\right| = \frac{\sqrt{R_L^2 + X_t^2}}{\sqrt{(R_L - \omega C R_L X_t)^2 + X_t^2}}$$

$$(R_L - \omega C R_L X_t)^2 + X_t^2 = \frac{1}{1.05^2}(R_L^2 + X_t^2)$$

$$C = \frac{1}{2\pi f R_L X_t} \cdot \left[R_L - \sqrt{\frac{1}{1.05^2}(R_L^2 + X_t^2) - X_t^2}\,\right] =$$

$$= \frac{1}{2\pi \cdot 60 \cdot 10 \cdot 1}\left[10 - \sqrt{\frac{1}{1.05^2}(10^2 + 1^2) - 1^2}\,\right] = \frac{0.4811}{2\pi \cdot 60 \cdot 10 \cdot 1} = 127.6\ \mu F$$

1.

Fall 2001 Prelim Exam Problem #2 Solution
(a) Series-Shunt Feedback Exam

Given Values:

$Rs := 20\cdot 10^3$ $\quad RC2 := 14.3\cdot 10^3$ $\quad R1 := 3\cdot 10^3$ $\quad R2 := 12\cdot 10^3$ $\quad RL := 10\cdot 10^3$

$VCC := 15$ $\quad VBE := 0.7$ $\quad VT := .025$ $\quad \beta := 200$ $\quad IC2 = 1\cdot 10^{-3}$

$IE3 = 2\cdot 10^{-3}$

DC Biasing Verification - Assume active region of operation.

$VC2 := VCC - RC2\cdot IC2$ $\qquad VC2 = 0.7$

$Vo := VC2 - VBE$ $\qquad Vo = 0$

Therefore all BJTs are in their active regions

(b) Open loop voltage gain. First find small signal model parameters.

$gm2 := \dfrac{IC2}{VT}$ $\qquad gm2 = 0.04$ $\qquad IC3 = \beta\cdot\dfrac{IE3}{\beta+1}$ $\qquad IC3 = 1.99\cdot 10^{-3}$

$r\pi2 := \dfrac{\beta}{gm2}$ $\qquad r\pi2 = 5\cdot 10^3$ $\qquad gm3 := \dfrac{IC3}{VT}$ $\qquad gm3 = 0.08$

$r\pi1 := r\pi2$ $\qquad r\pi3 := \dfrac{\beta}{gm3}$ $\qquad r\pi3 = 2.513\cdot 10^3$

Begin analysis with first stage. Find gain from node 1 to node 2 and from node 2 to node 3. The loading of feedback network on input is:

$RB2 := \dfrac{1}{\left[\dfrac{1}{R1}+\dfrac{1}{R2}\right]}$ $\qquad RB2 = 2.4\cdot 10^3$

The loading of the feedback on the output is:

$RLfb := R2 + R1$

$RLfb = 1.5\cdot 10^4$

$RinE2 := \dfrac{r\pi2 + RB2}{\beta+1}$ $\qquad RinE2 = 36.816$

$RinB1 := r\pi1 + (\beta+1)\cdot RinE2$ $\qquad RinB1 = 1.24\cdot 10^4$ $\qquad R_i := RinB1$ $\qquad R_i = 1.24\cdot 10^4$

(c) $Av12 = \dfrac{RinB1}{(Rs + RinB1)}$ $\qquad Av12 = 0.383$

Now for the first stage (Q1) voltage gain which is a common collector configuation.

$Av23 := (\beta+1)\cdot\dfrac{RinE2}{RinB1}$ $\qquad Av23 = 0.597$

Now for the second stage (Q2) gain which is a common base configuration.

$RL3 := \dfrac{1}{\left[\dfrac{1}{RL}+\dfrac{1}{RLfb}\right]}$ $\qquad RL3 = 6\cdot 10^3$

$RinB3 := r\pi3 + (\beta+1)\cdot RL3$ $\qquad RinB3 = 1.209\cdot 10^6$

5, cont

$$RL2 := \frac{1}{\left[\frac{1}{RC2} + \frac{1}{RinB3}\right]}$$ $RL2 = 1.413 \cdot 10^4$

$$Av34 := \left[\frac{\beta}{\beta + 1}\right] \cdot \frac{RL2}{RinE2}$$ $Av34 = 381.967$

Now for the third stage (Q3) gain which is a common collector configuration.

$$Av45 := (\beta + 1) \cdot \frac{RL3}{RinB3}$$ $Av45 = 0.998$

$Av25 := Av23 \cdot Av34 \cdot Av45$ $Av25 = 227.474$ Open Loop voltage gain

$Av15 := Av12 \cdot Av23 \cdot Av34 \cdot Av45$ $Av15 = 87.058$

(d) Now determine the feedback factor - use BetaF so as to not be confused with transistor betas.

$$\beta F := \frac{R1}{(R1 + R2)}$$ $\beta F = 0.2$

$D := 1 + Av25 \cdot \beta F$ $D = 46.495$

(e) Find the voltage gain with feedback

$$Avf25 := \frac{Av25}{1 + Av25 \cdot \beta F}$$ $Avf25 = 4.892$

(f) Find the input resistace of the amplifier with feedback

$RinB1f := RinB1 \cdot D$ $RinB1f = 5.765 \cdot 10^5$

(g) Find the output resistance without feedback, Ro.
First we look back into the emitter of Q3.

$$RinE3 := \frac{r\pi 3 + RC2}{\beta + 1}$$ $RinE3 = 83.644$

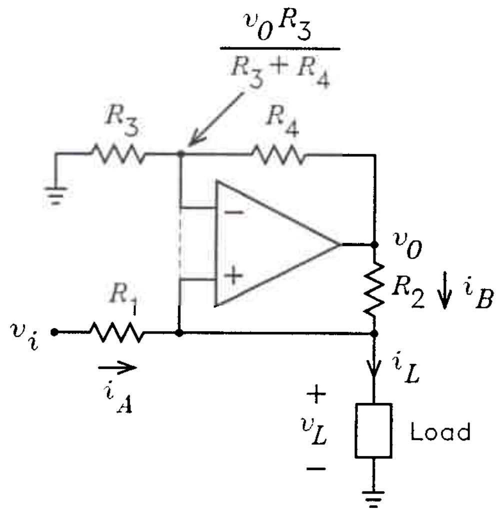
$$Ro := \frac{1}{\left[\frac{1}{RLfb} + \frac{1}{RinE3}\right]}$$ $Ro = 83.18$

(h) Lastly, find the output resistance with feedback, Ro.

$$Rof := \frac{Ro}{D}$$

$Rof = 1.789$

Solution:



The load current is given by

$$i_L = i_A + i_B \qquad \Longrightarrow \qquad i_L = \frac{v_I - v_L}{R_1} + \frac{v_O - v_L}{R_2}$$

But there is a virtual short between the two op-amp inputs so that v_L is given by

$$v_L = \frac{v_O R_3}{R_3 + R_4}$$

Thus the equation for i_L becomes

$$i_L = \frac{v_I - \dfrac{v_O R_3}{R_3 + R_4}}{R_1} + \frac{v_O - \dfrac{v_O R_3}{R_3 + R_4}}{R_2} = \frac{v_I}{R_1} - \frac{1}{R_1}\frac{v_O R_3}{R_3 + R_4} + \frac{1}{R_2}\frac{v_O R_4}{R_3 + R_4}$$

This is independent of v_O, and thus v_L, if

$$\frac{1}{R_1}\frac{v_O R_3}{R_3 + R_4} = \frac{1}{R_2}\frac{v_O R_4}{R_3 + R_4}$$

which leads to the condition

$$\frac{R_3}{R_1} = \frac{R_4}{R_2}$$

In which case, i_L is given by

$$i_L = \frac{v_I}{R_1}$$

Problem # 1 Solution

Method 1:

$$E_f - E_i = -kT \ln\left[\frac{N_A(x)}{n_i}\right]$$

$$= -kT \ln\left[\frac{1e16\, e^{(x/5\mu m)}}{1e10}\right]$$

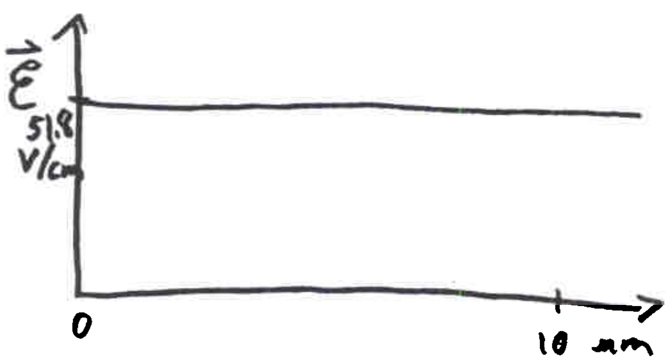
$$= -kT \ln[1e6] - kT \ln\left[e^{x/5\mu m}\right]$$

$$E_i = E_f + kT \ln[1e6] + \left(\frac{x\, kT}{5\mu m}\right)$$

$$\vec{\mathcal{E}} = \frac{1}{q}\frac{dE_i}{dx} = \frac{dE_F}{dx} + \frac{kT}{5\mu m}$$

But in equilibrium the Fermi energy is Flat (i.e. no net current flow)

$$\vec{\mathcal{E}} = \frac{kT}{q}\,\frac{1}{0.0005} = \frac{0.0259V}{0.0005} = 51.8\ V/cm$$



7, cont

Method 2:

In equilibrium, $J_p = J_n = 0$

$$J_p = \underbrace{q \mu_p p \vec{\mathcal{E}}}_{\text{Drift Current}} - \underbrace{q D_p \nabla p}_{\text{Diffusion Current}} = 0$$

$$\vec{\mathcal{E}} = \frac{q D_p \nabla p}{q \mu_p p} = \underbrace{\frac{kT}{q}}_{\text{Einstein Relationship:}} \frac{\nabla p}{p}$$

Einstein Relationship:

$$\frac{kT}{q} = \frac{D_p}{\mu_p}$$

$$\mathcal{E} = \frac{kT}{q} \frac{\left[\frac{1}{5\mu m}\left(1e16\, e^{x/5\mu m}\right)\right]}{\left(1e16\, e^{x/5\mu m}\right)}$$

$$= \frac{kT}{q} \frac{1}{0.0005\,cm}$$

$$= 51.8\ V/cm$$

Plot is the same as previous page.

Problem #3 - Solution

Using minority carrier diffusion eq$^{\underline{n}}$ for holes:

$$\frac{\partial \Delta p_n}{\partial t} = D_p \frac{\partial^2 \Delta p_n}{\partial x^2} - \frac{\Delta p_n}{\tau_p} + G_L$$

Situation: steady-state, no concentration gradient

$$\Rightarrow \emptyset = -\frac{\Delta p_n}{\tau_p} + G_L$$

$$\Rightarrow \Delta p_n = \Delta n_n = G_L \tau_p = 10^{15}\, cm^{-3}$$

(a) $n = n_o + \Delta n \simeq \boxed{10^{17}\, cm^{-3}}$

$$p = p_o + \Delta p = \frac{n_i^2}{n_o} + \Delta p \cong \boxed{10^{15}\, cm^{-3}}$$

8, cont

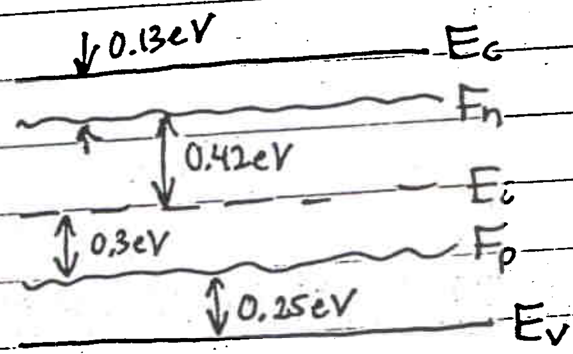
Problem #3 (cont.)

(b) $n = n_i e^{(F_n - E_i)/kT} \Rightarrow F_n - E_i = kT \ln\left(\frac{n}{n_i}\right) = 0.42eV$

$p = n_i e^{(E_i - F_p)/kT} \Rightarrow E_i - F_p = kT \ln\left(\frac{p}{n_i}\right) = 0.3eV$

$\Rightarrow F_n - F_p = (F_n - E_i) + (E_i - F_p) = \boxed{0.72eV}$

(c) Not to ~~sc~~ scale:



Solution

(9)

Consider the following MIPS program subroutine.

address	label	instruction
1000	start:	addi $2, $0, 1
1004	loop:	slti $3, $1, 2
1008		bne $3, $0, skipl
1012		mul $2, $1, $2
1016		addi $1, $1, -1
1020		beq $0, $0, loop
1024	skipl:	jr $31

Part A If $1 = 5 when start is called, how many times is the `mul` instruction at address 1012 executed?

number of executed `mul` instructions: **4**

Part B What is the branch offset (in bytes) for the `beq` instruction at address 1020?

branch offset (in bytes): **-5 * 4 = -20**

Part C If $31 contains the value 5000 (in decimal), what is the address of the `jal` instruction that called this subroutine?

`jal` address (in decimal): **4996**

Part D After this subroutine completes, which register contains the results computed by this subroutine?

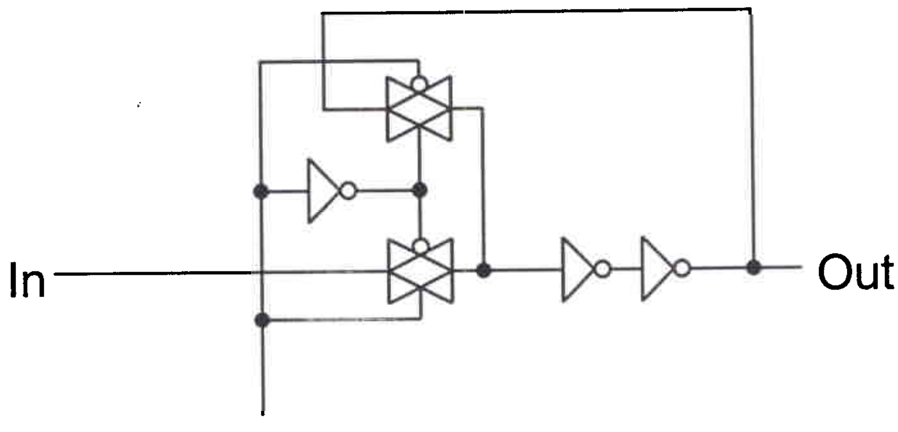
result register: **120**

Part E What mathematical function does this program subroutine accomplish?

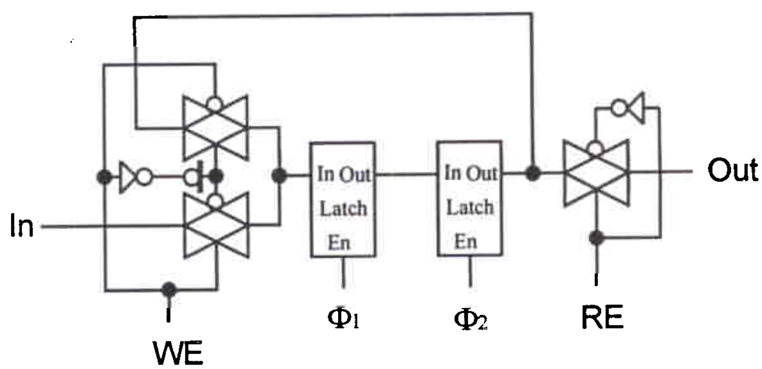
This fragment computes: **factorial of register one ($1!)**

Solution (10)

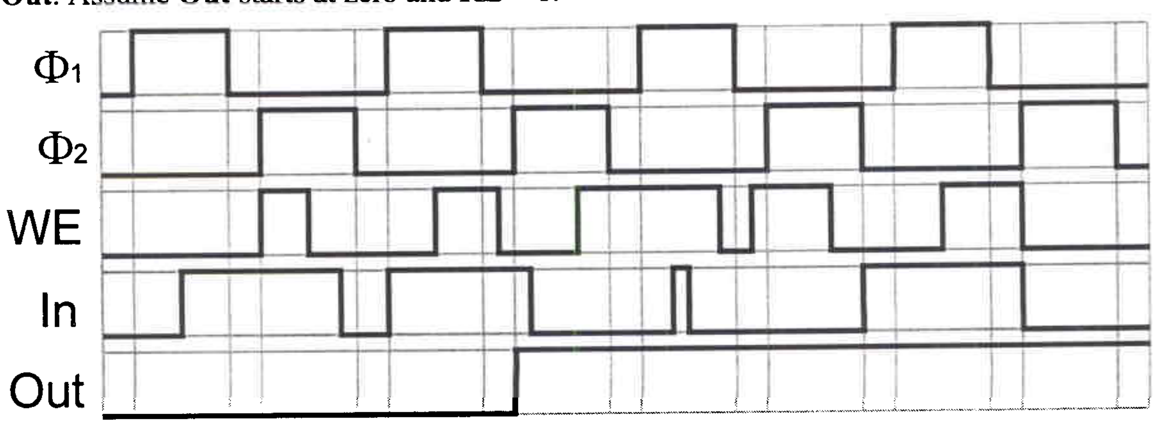
Part A Implement a transparent latch using only pass gates and inverters. Label the inputs **In** and **En**, and the output **Out**.



Part B Implement a register with write enable and read enable using transparent latches, pass gates, and inverters. Use an icon for the transparent latches. Label the inputs **In**, **WE, RE,** Φ_1, Φ_2 and the output **Out**.



Part C Assume the following signals are applied to your register. Draw the output signal **Out**. Assume **Out** starts at zero and **RE** = 1.



Solution (11)

1. Consider a byte addressable memory hierarchy with 32 bit addressing and L1 and L2 caches, both with block size of 4 words (16 bytes). Suppose the 32 Kilobyte L1 cache is direct mapped and write through, and the 512 Kilobyte L2 cache is 4-way set associative with a random replacement policy. Also assume that both caches allocate on either a read miss or a write miss. Show that for this configuration, if a block is in L1 it is also in L2, and if it is modified in L1, it is modified in L2 (This property is known as multi-level inclusion, and implies that one only need check L2 to determine the status of a block). Hint: Assume that the property holds initially (for example, when both caches are empty, the property holds trivially) and show that for any possible event that can occur with respect to a block in either cache, the property still holds after the event.

Consider the L1 cache: It has 2^{15} bytes and 2^{11} blocks and sets (since it is direct mapped. The address $(a_{31} \ldots a_0)$ is partitioned as (from lsb to msb) 4 bits of offset $(a_3 - a_0)$, 11 bits of index $(a_{14} - a_4)$, and 17 bits of tag $(a_{31} - a_{15})$.

Similarly, the L2 cache has 2^{19} bytes, 2^{15} blocks, and 2^{13} sets (since there are 4 blocks per set. The address for the L2 is partitioned as 4 bits of offset $(a_3 - a_0)$, 13 bits of index $(a_{16} - a_4)$, and 15 bits of tag $(a_{31} - a_{17})$.

To show that a block present in L1 is also present (in the same state) in L2, we have the following steps:

1. A miss in L1 only will replace a block in L1 from a block present in L2. Inclusion is preserved.
2. A write in L1 is written through to L2 and inclusion is preserved.
3. A miss in L1 and L2 is allocated in both L1 and L2 and inclusion is preserved for the address which missed.
4. A miss in L1 and L2 require a victim in both L1 and L2. Since for each address the L1 index is a subset of the L2 index, the L2 victim which is selected from the set specified by the L2 index, must map to the block specified by the L1 index. That is all possible victims map to the same block in L1. Consequently, the victim in L2 cannot remain in L1, and inclusion is preserved.

12

Virtual Memory Systems

Assume a 16-entry direct-mapped TLB is used in a system with 32-bit virtual byte addresses, 24-bit physical memory byte addresses, and 4096 byte pages. For the sequence of virtual addresses shown below in hex, state whether each address causes a TLB hit or miss and show the physical address generated. Portions of the initial contents of the TLB and the page table are given. Assume all listed page table entries are valid. Show the final contents of the TLB after this sequence of addresses is accessed.

Virtual address sequence

Page offset

Address	*Hit/Miss*	*Physical Address*
00000171	H	200 171
00022CE2	H	A00 CE2
00023AE3	M	E20 AE3
00001012	M	F00 012
00000EE1	H	200 EE1
00020153	M	720 153

TLB Tag

TLB Block Address

TLB Initial Contents

Block	*Tag*	*Data*
0	~~0000~~ 0002	~~200~~ 720
1	~~0002~~ 0000	~~C00~~ F00
2	0002	A00
3	~~0020~~ 0002	E20

TLB Final Contents

Block	*Tag*	*Data*
0	0002	720
1	0000	F00
2	0002	A00
3	0002	E20

Page Table Initial Contents

Virtual Page #	*Data*
00000	200
00001	F00
00002	550
00003	300
...	
00020	720
00021	C00
00022	A00
00023	E20

Questionis Preliminari (de Systemibus et Controlibus)

PROVE THAT IF INTEGRAL FEEDBACK IS USED ON A STRICTLY PROPER SYSTEM, THE CENTER OF GRAVITY OF THE CLOSED LOOP POLES ARE INDEPENDENT OF THE GAIN.

Solution:

Let the transfer function be $H(s) = \frac{b(s)}{a(s)}$. By assumption,

$$\begin{aligned} a(s) &= s^n + a_1 s^{n-1} + \cdots \\ b(s) &= b_1 s^{n-1} + b_2 s^{n-2} + \cdots \end{aligned}$$

With input $u(t) = v(t) - \int^t ky(\tau)\, d\tau$, the closed loop has transfer function

$$H_k(s) = \frac{sb(s)}{sa(s) + kb(s)}$$

The characteristic polynomial is

$$sa(s) + kb(s) = s^{n+1} + a_1 s^n + (a_2 + b_1)s^{n-1} + \cdots \qquad (1)$$

Now recall that if a system has n poles $\mu_1, \ldots, \mu_n$, then its characteristic polynomial is

$$\alpha(s) = (s - \mu_1) \cdots (s - \mu_n) = s^n - (\mu_1 + \ldots + \mu_n)s^{n-1} + \ldots.$$

Thus the coefficient of the $(n-1)$st term is

$$\alpha_1 = -(\mu_1 + \ldots + \mu_n) = -n\ \{\text{CG of poles}\}.$$

Applying to the $(n+1)$st order closed loop characteristic equation (1), we get thus for the closed loop system, the center of gravity the poles

$$-\frac{a_1}{n+1},$$

which is independent of k. **QED**

Discussion:

The problem requires

1. understanding the relation between poles and coefficients in characteristic polynomial
2. knowledge of integral feedback
3. knowledge of transfer function reduction
4. knowledge of basic idea of root locus
5. knowledge of the meaning of "CG of the poles"
6. knowledge of the notion "strictly proper"

14

Solution

a

If we let x_1, x_2 denote the position and velocity of the mass respectively, we can let the state be $x = (x_1, x_2)^T$. We get

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

b

The characteristic equation of the unforced system is

$$r^2 + \frac{k}{m} = 0,$$

with imaginary roots

$$r = \pm i\sqrt{\frac{k}{m}}.$$

Hence $y(t)$ is given by

$$y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right),$$

with

$$c_1 = y_0$$
$$c_2 = \sqrt{\tfrac{m}{k}}\dot{y}_0.$$

c

Two poles in -1 gives the desired characteristic polynomial as

$$(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1.$$

Using state feedback $u(t) = (f_1, f_2)x(t)$ gives the closed-loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -k/m + f_1/m & f_2/m \end{pmatrix} x.$$

The characteristic polynomial is

$$\det\left[\begin{pmatrix} \lambda & -1 \\ k/m - f_1/m & \lambda - f_2/m \end{pmatrix}\right] = \lambda^2 - \frac{f_2}{m}\lambda + \frac{k - f_1}{m}.$$

Identification of the coefficients gives that

$$-f_2/m = 2$$
$$(k - f_1)/m = 1,$$

i.e. $f_1 = k - m$ and $f_2 = -2m$ gives the desired pole placement.

14, cont

d

We let $u = gy$, which gives the closed-loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -k/m + g/m & 0 \end{pmatrix} x,$$

with characteristic polynomial

$$\lambda^2 + \frac{k-g}{m}.$$

But, since we demand that the coefficient in front of λ is 2 we can not place the poles in -1 no matter what g is.

Solutions for 3075 Prelim problems for 2001-2002

Problem 1:

a) U is a Gaussian RV because it is the sum of two jointly distributed Gaussian RVs. Therefore, to specify the PDF of U, we need only the mean and variance of U.

$$E(U) = E([X,Y])\begin{bmatrix} 4 \\ -3 \end{bmatrix} = [0 \quad -3]\begin{bmatrix} 4 \\ -3 \end{bmatrix} = 9$$

$$Var(U) = E\left([U - E(U)]^2\right) = E\left\{\left(A[W - E(W)]^T\right)^2\right\} = AE\left([W - E(W)]^T[W - E(W)]\right)A^T$$

$$= AC_W A^T = [4,-3]\begin{bmatrix} 2.0 & -0.4 \\ -0.4 & 1.0 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = 50.6$$

Therefore, the PDF of U is

$$f_U(u) = \frac{1}{\sqrt{2\pi 50.6}}\exp\left(\frac{(u-9)^2}{101.2}\right) \qquad \text{for all } u$$

(15) ✓

b) Recall $Cov(X,Y) = E(XY) - \mu_X \mu_Y$. Then,
$E(XY) = Cov(X,Y) + \mu_X \mu_Y = -0.4 + 0 = -0.4$

Problem 2:

For 1<Y<3, there will be one solution, X=(3-Y)/2. The derivative of the function in this region is -2. Therefore, for 1<Y<3,

$$f_Y(y) = \frac{f_X\left(\frac{3-y}{2}\right)}{|-2|} = \frac{3\exp\left(-3\left[\frac{3-y}{2}\right]\right)}{2} = \frac{3}{2}\exp\left(\frac{3}{2}[3-y]\right)$$

The PDF will be zero everywhere else except at Y=1, where there is an impulse of area

$$\int_1^{+\infty} 3\exp(-3x)dx = -\exp(-3x)\Big|_1^{+\infty} = \exp(-3)$$

Therefore, the PDF of Y is

$$f_Y(y) = \begin{cases} \frac{3}{2}\exp\left(\frac{3}{2}[3-y]\right) + \exp(-3)\delta(y-1) & 1 \le y > 3 \\ 0 & otherwise \end{cases}$$

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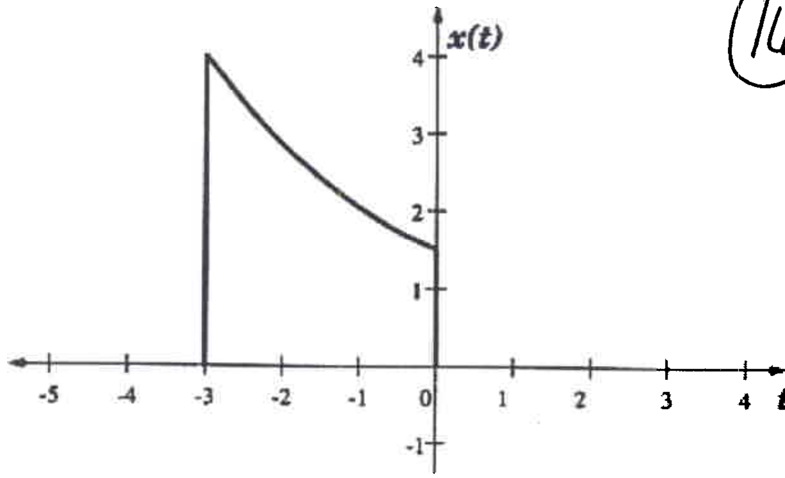
(a)



Where $x(t) = 4e^{-(t+3)/3}\,[u(t+3) - u(t)]$.

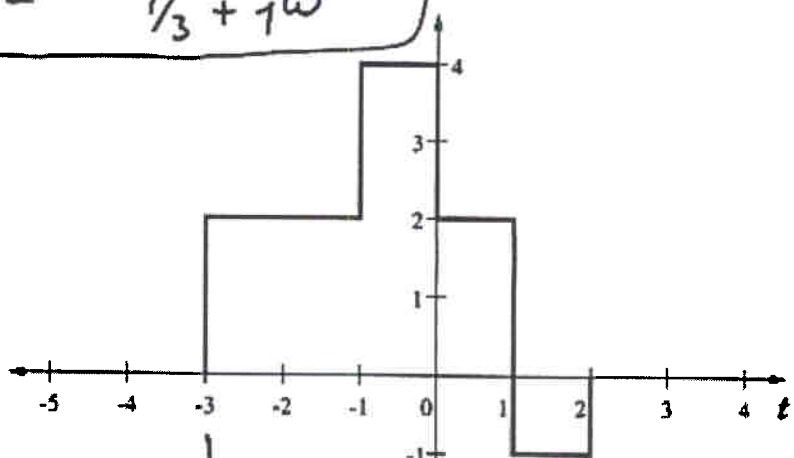
$$x(t) = 4e^{-(t+3)/3}\,u(t+3) - 4e^{-(t+3)/3}\,u(t)$$

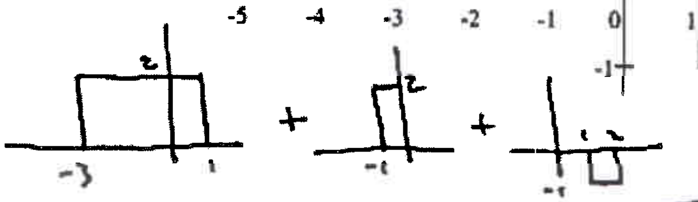
$$= 4e^{-\frac{(t+3)}{3}}\,u(t+3) - 4e^{-1}e^{-t/3}\,u(t)$$

$$X(j\omega) = \frac{4}{1/3 + j\omega}e^{j3\omega} - \frac{4e^{-1}}{1/3 + j\omega}$$

$$\boxed{X(j\omega) = \frac{4\left(e^{-j3\omega} - e^{-1}\right)}{1/3 + j\omega}}$$

(b)



$x(t) =$  $=$

$$\boxed{X(j\omega) = 2\frac{\sin(2\omega)}{\omega/2}e^{j\omega} + 2\frac{\sin(\omega/2)}{\omega/2}e^{j\frac{\omega}{2}} - \frac{\sin(\omega/2)}{\omega/2}e^{-j\frac{3\omega}{2}}}$$

Note: there are many possible answers.

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Prelim Solution **Data Structures/Algorithms**

Consider the following program written in pseudocode:

```
FOO(u, v)
    if v=0
    then return u
    else return FOO(v, u mod v)
```

Part A. How many stack activation frames are created in executing `FOO(18, 12)`?

________________3________________**stack activation frames.**

Part B. Describe in one sentence what `FOO` computes.

FOO computes the greatest common divisor of its two integer inputs u and v.

Part C. Give an expression for the number of nodes n in a balanced tree having a constant branching factor of b and a height of h.

$$\mathbf{n} = \sum_{i=0}^{h} b^i$$

18

Computer Science Prelim Question

Consider the following set of processes, their arrival times, and the CPU times they require.

Process	*Arrival Time*	*CPU Time*
P0	0 ms	4 ms
P1	1 ms	3 ms
P2	3 ms	8 ms
P3	7 ms	6 ms
P4	9 ms	8 ms
P5	14 ~~13~~ ms	2 ms

Show the schedule generated by a round robin scheduling algorithm with a time slice of 2 ms and calculate the average waiting time of a process for this schedule.

0	2	4	6	8	9	11	13
P0	P1	P0	P2	P1	P3	P2	P4

P0 completes (at 6)

P1 completes (at 9)

15	17	19	21	23	25	27	29	31
P3	P2	P5	P4	P3	P2	P4	P4	

P5 completes (at 21)

P3 completes (at 25)

P2 completes (at 27)

P4 completes (at 31)

$WT_0 = 6-4 = 2\ ms$

$WT_1 = 9-3-1 = 5\ ms$

$WT_2 = 27-8-3 = 16\ ms$

$WT_3 = 25-6-7 = 12\ ms$

$WT_4 = 31-8-9 = 14\ ms$

$WT_5 = 21-2-14 = 5\ ms$

$A.W.T. = \frac{2+5+16+12+14+5}{6}$

$A.W.T. = 9\ ms$

(19)

Question 1 – Nyquist's Law

According to Nyquist, if a digital encoder can NRZ encode 32 levels per time slot of 10 microseconds, what is the:

[5] Bits per symbol.

[100,000 /s] Baud Rate (Baud/s).

[500,000 b/s] Capacity (bits per second).

[50,000 Hz] Bandwidth required (Hz).

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Problem # 6 Solution

This problem consists of **two parts,** part (a) and part (b).

An initially bare silicon wafer is to be oxidized in a dry oxygen ambient. Assume that the oxidation follows the standard Deal-Grove model with the following constants:

Temperature (°C)	A (microns)	B (microns^2 per hour)
1000	0.165	0.0117
1100	0.090	0.027

(a) At a temperature of 1000 °C, calculate the time necessary to grow an oxide of 0.2 micron thickness on the wafer

(b) After the initial oxidation, the wafer is put back into the furnace at a temperature of 1100 °C for 1 hour. Calculate the thickness of the oxide after this second oxidation. (Hint: after the second oxidation, the oxide should be thicker than 0.2 microns!!)

Solution:

X delete

give this in problem statement

Deal-Grove Model:

~~$d_0^2 + A\,d_0 = B\,(t + \tau)$ and $\tau = (d_i^2 + A\,d_i)/B$,~~
where d_0 is oxide thickness and d_i is initial oxide thickness

(a) at 1000 C, ~~wafer is initially bare so $\tau = 0$ and~~: $\tau = 0.37$ hours

~~$t = (0.2^2 + (.165)*0.2))/(0.0117) = 6.2$ hour~~ $t = \frac{x_o^2 + Ax_o}{B} - \tau = \frac{0.2^2 + (.165)(0.2)}{0.0117} - 0.37$

$\boxed{t = 5.87 \text{ hours}}$

(b) now $d_i = 0.2$ microns and $T = 1100$ C and $t = 1$ hour so:

$\tau = (0.2^2 + (0.09)*0.2)/0.027 = 2.1$ hour
and $d_0^2 + 0.09\,d_0 = 0.027*(1 + 2.1)$
or:
$d_0^2 + 0.09\,d_0 - 0.0837 = 0$
Using quadratic formula:
$d_0 = 0.248$ micron

Group Velocity

The group velocity is defined by

$$v_g = \frac{d\omega}{dk}$$

Using

$$k = \frac{n\omega}{c}$$

it is evaluated to be

$$v_g = \frac{1}{\frac{dk}{d\omega}} = \frac{1}{\frac{d(\frac{n\omega}{c})}{d\omega}} = \frac{1}{\left[\frac{dn}{d\omega}(\frac{\omega}{c}) + \frac{n}{c}\frac{d\omega}{d\omega}\right]} = \frac{c}{\left(\frac{dn}{d\omega}\omega + n\right)}$$

Further

$$\frac{dn}{d\omega} = \frac{dn}{d\lambda}\frac{d\lambda}{d\omega}$$

$$\lambda = fc$$

$$\frac{d\lambda}{d\omega} = \frac{c}{2\pi}$$

$$\frac{dn}{d\omega}\omega = \frac{dn}{d\lambda}\frac{c}{2\pi}\frac{2\pi\lambda}{c} = \frac{dn}{d\lambda}\lambda$$

and so

$$v_g = \frac{c}{\left(n - \lambda\frac{dn}{d\lambda}\right)}.$$

22

Two Lens Combination

Two thin positive lenses L_1 and L_2 with focal lengths f_1 and f_2 respectively are separated by a distance d. The separation distance is less than the sum of their focal lengths. An object is placed in front of L_1 a distance $u_1 > f_1$.

a) What is the position, of the image?
b) What is the magnification,?
c) What is the orientation of the image?

For a) and b) leave your answer in terms of the focal lengths, the object distance and the separation d

Using Gauss' lens equation , the relations for L_1 is:

a)

$$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1}$$

where v_1 is the image distance.

Solving for the image distance gives $v_1 = u_1 f_1/(u_1 - f_1)$.

The position of the object for L_2 is given by:

$$v_2 = \frac{u_2 f_2}{u_2 - f_2} = \frac{(d - v_1) f_2}{(d - v_1 - f_2)}$$

Substituting in for v_1 gives:

$$v_2 = \frac{f_2\left[d - f_1 u_1/(u_1 - f_1)\right]}{\left[d - f_1 u_1/(u_1 - f_1) - f_2\right]}$$

b) The magnification of the combination is given by:

$$m = m_1 m_2$$

$$m = \left(\frac{f_1}{u_1 - f_1}\right)\underbrace{\left(\frac{f_2}{d - v_1 - f_2}\right)}_{\text{negative}}$$

c.) The image is inverted.

PROBLEM 23 (Specialized: Bioengineering) Code Number: ______

Show your work. No credit will be given if you only provide the answer. Give units where appropriate.

Make the conventional assumption throughout that the transmembrane voltage is the intracellular potential minus the extracellular potential and that positive ionic currents flow from *i(intracellular)* to *o(extracellular)*.

Consider the axon of a giant sea worm living at the bottom of the ocean in the Marianas Trench. The axon of this giant sea worm has the following intracellular and extracellular concentrations: $c_{o,K}$ = 2.2mM, $c_{i,K}$ = 100mM, $c_{o,Na}$ = 109mM, $c_{i,Na}$ = 16mM $c_{o,Cl}$ = 80mM, $c_{i,Cl}$ = 5mM. The resting potential is -97mV. Note: K, Na, Cl = potassium, sodium and chloride ions respectively

a) (3 pts) Find the Nernst potential associated with each of these ions.
b) (3pts) Consider the axon at rest. In what direction are the ions flowing? Are any ions in equilibrium or nearly in equilibrium? If so, which ones?
c) (4pts) From the Goldman-Hodgkin-Katz equation provided below, calculate the ratio of the permeabilities of the membrane to the ions which are not close to being in equilibrium. For example. P_K:P_{Na} if Cl is close to equilibrium. That is to say, neglect the participation of the ion which is closest to equilibrium.

$$V_{rest} = 25.8mV \ln \frac{P_K c_{o,K} + P_{Na} c_{o,Na} + P_{Cl} c_{o,Cl}}{P_K c_{i,K} + P_{Na} c_{i,Na} + P_{Cl} c_{i,Cl}}$$

a) $V_K = \frac{RF}{F} \ln\left(\frac{C_{o,K}}{C_{i,K}}\right) = 25 \ln\left(\frac{2.2mM}{100mM}\right) = -95.4mV$

$V_{Na} = 25 \ln\left(\frac{109mM}{16mM}\right) = 48mV$

$V_{Cl} = \frac{25}{-1} \ln\left(\frac{80mM}{5mM}\right) = -69.7mV$

b) $V_{rest} = -97mV$

K: $V_{rest} - V_K = -1.6mV \Rightarrow$ very little driving force, i.e. ~ in equilibrium

$I_K = g_K(V_{rest} - V_K) < 0 \Rightarrow$ K flows slightly o to i

Na: $I_{Na} = g_{Na}(V_{rest} - V_{Na}) = g_{Na}(-97 - 48) = g_{Na}(-145mV) < 0 \Rightarrow$ o to i

Cl: $I_{Cl} = g_{Cl}(V_{rest} - V_{Cl}) = g_{Cl}(-97 - -69.7) = g_{Cl}(-27.3mV) < 0$ $\Rightarrow$ o to i

K is nearly in equilibrium

c) K ~ inequil, so

$$V_{rest} = 25.8 \ln\left[\frac{P_{Na} C_{o,Na} + P_{Cl} C_{i,Cl}}{P_{Na} C_{i,Na} + P_{Cl} C_{o,Cl}}\right] = -97 mV$$

$$\frac{P_{Na} C_{o,Na} + P_{Cl} C_{i,Cl}}{P_{Na} C_{i,Na} + P_{Cl} C_{o,Cl}} = e^{-97/25.8} = 2.33\times10^{-2}$$

$$P_{Na}(C_{o,Na} - 2.33\times10^{-2} C_{i,Na}) = P_{Cl}(2.33\times10^{-2} C_{o,Cl} - C_{i,Cl})$$

$$\left|\frac{P_{Na}}{P_{Cl}}\right| = \left|\frac{(2.33\times10^{-2}\, 80 mM - 5 mM)}{(-2.33\times10^{-2}\, 16 mM + 109)}\right| = \left|\frac{+3.14}{108.6}\right| = 2.89\times10^{-2}$$

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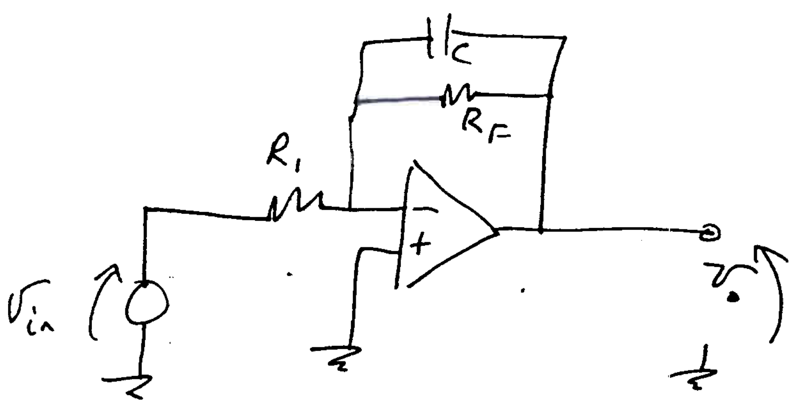


$R_{input} = 10\,k\Omega$

$\tau = 1.2\,ms$

$R_2 = R_F$

$$\frac{V_o(s)}{V_i(s)} = \frac{V_o(j\omega)}{V_i(j\omega)} = -\left(\frac{\dfrac{R_F \frac{1}{j\omega C}}{R_F + \frac{1}{j\omega C}}}{R_1}\right) = \frac{-1}{j\omega C R_1} \quad -\frac{R_F}{R_1}$$

$\tau = C R_F$

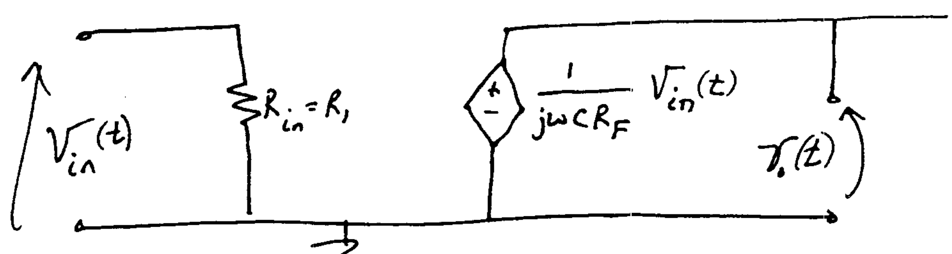
$$\frac{V_o}{V_I} = \text{DC Gain} = \left|-\frac{R_2}{R_1}\right| = \left|\frac{R_F}{R_1}\right| = 100 \Rightarrow R_F = R_1(100)$$

$$\boxed{R_{input} = R_1 = 10\,K\Omega} \Rightarrow \boxed{R_F = R_2 = 1\,M\Omega}$$

If $\tau = 1.2 \times 10^{-3}$ seconds $= C R_F$

$$1.2 \times 10^{-3} = (1 \times 10^{6})(C)$$

$$\boxed{C = 1.2 \times 10^{-9} F = 1.2\,nF}$$



$$\boxed{\omega_{3dB} = \frac{1}{R_2 C} = \frac{1}{R_F C} = 833\,Hz}$$